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De curvis tractoriis compositis

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DE CVRVIS TRACTORIIS COMPOSITIS.

Auctore
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§. 1.

Quando filo, cuius alter terminus ſuper plano horizontali per datam viam protrahitur, duo pluraue corpuſcula fuerint alligata, ita vt ſingula per curuas peculiares procedant, iſtae curuae *Tractoriae compoſitae* ſunt appellatae, quas hic ſimili modo, quo nuper Tractorias ſimplices tractaui, accuratius inveſtigare conſtitui.

§. 2. Primum autem hic obſeruo, ſi hanc quaeſtionem ſecundum principia mechanica, quorſum ea vtique proprie eſt referenda, euoluere vellemus, tunc quidem facile ad formulas differentiales ſecundi gradus perduceremur, quas autem nullo adhuc modo ob defectum Analyſeos reſoluere licet. Hinc iſtam quaeſtionem a Mechanica ad puram Geometriam ſimili modo ſum translaturus, quo Geometrae Tractoriam vulgarem contemplari ſunt ſoliti. Loco ſcilicet verorum principiorum motus hic ſubſtituam hanc Hypotheſin: quod viribus ſollicitantibus non accelerationes quibus ſingula corpuſcula promouentur, ſed ipſa ſpatiola tempuſculo minimo deſcripta, ſint proportionalia, cuiusmodi motum eſſent ſecutura, ſi quovis momento motus iam genitus ſubito deſtrueretur et continuo de nouo generari deberet, quemadmodum vere eueniret, ſi frictio eſſet infinite magna. Iam olim quidem a Marchione Hoſpitalio in Analyſi infinitorum tangentes huiusmodi curuarum definitae reperiuntur; non autem memini vtrum prorſus eadem Hypotheſi ſit vſus. Ceterum autem iſtas curuas accuratius hic determinare conabor, quo magis pateat, quantis difficultatibus huiusmodi quaeſtiones, quae primo intuitu faciles videantur, adhuc ſint obuolutae.

Problema I.

Si filum duobus corpuſculis A et B fuerit onuſtum, eiusque terminus R ſuper plano horizontali iuxta lineam rectam I O protrahatur, inueſtigare ambas curuas, quas haec duo corpuſcula deſcribent. Tab. I. Fig. 7.

Solutio.

§. 3. Elapſo tempore t filum cum corpuſculis iam perductum ſit in ſitum A B R, ſintque fili portiones $AB = a$ et $BR = b$, dum litterae maiuſculae A et B exprimunt maſſam vtriusque corporis. Hinc ad rectam I O, tanquam ad axem, ducantur perpendicula A P et B Q, ponanturque coordinatae vtriusque curuae $IP = x$, $PA = y$ et $IQ = x'$, $QB = y'$; pro puncto R autem ſit ſpatium $IR = x''$, exiſtente $y'' = 0$. Praeterea vero vocemus angulos $PAB = p$ et $QBR = q$, ac manifeſtum eſt fore $IQ = x' = x + a\,\text{ſin.}\,p$ et $QB = y' = y - a\,\text{coſ.}\,p$; tum vero $x'' = x + a\,\text{ſin.}\,p + b\,\text{ſin.}\,q$ et $y'' = y - a\,\text{coſ.}\,p - b\,\text{coſ.}\,q = 0$. Hinc ergo ſumtis differentialibus erit

$$\partial x' = \partial x + a\,\partial p\,\text{coſ.}\,p,$$
$$\partial y' = \partial y + a\,\partial p\,\text{ſin.}\,p,$$
$$\partial x'' = \partial x + a\,\partial p\,\text{coſ.}\,p + b\,\partial q\,\text{coſ.}\,q \text{ et}$$
$$\partial y'' = \partial y + a\,\partial p\,\text{ſin.}\,p + b\,\partial q\,\text{ſin.}\,q = 0.$$

§. 4. Cum nunc corpuſcula alias vires non ſuſtineant, niſi quibus filum tenditur, quandoquidem ratio frictionis tanquam infinite ſpectatae iam in noſtra hypotheſi ſtabilita inuoluitur, ſit tenſio portionis $AB = T$, portionis autem $BR = T'$, quibus poſitis corpuſculum A in directione AB ſollicitatur vi $= T$, quae ſecundum coordinatas reſoluta praebet vim ſecundum $IP = T \text{ coſ. } p$ et ſecundum directionem $AP = - T \text{ coſ. } p$; alterum vero corpuſculum duas ſuſtinet vires, alteram ſecundum $BA = T$, alteram vero ſecundum $BR = T'$, ex quarum reſolutione naſcuntur: 1°) vis ſecundum $IQ = - T \text{ ſin. } p + T' \text{ ſin. } q$ et 2°) ſecundum QB vis $= + T \text{ coſ. } p - T' \text{ coſ. } q$. His igitur viribus proportionalia ſunt ſpatiola tempuſculo ∂t percurſa ſecundum easdem directiones, vel potius ipſe motus, qui oritur ſi ſpatiola illa per maſſas vtriusque corpuſculi multiplicentur, quandoquidem maſſarum ratio hic inprimis eſt habenda.

§. 5. Quod ſi ergo motus vtriusque corpuſculi etiam ſecundum directiones coordinatarum reſoluatur, formulae viribus proportionales erunt $A . \frac{\partial x}{\partial t}$ et $A . \frac{\partial y}{\partial t}$ pro corpuſculo A: at $B . \frac{\partial x'}{\partial t}$ et $B \frac{\partial y'}{\partial t}$ pro corpuſculo B, hincque nanciſcimur ſequentes quatuor aequationes:

$$\text{I. } \frac{A \partial x}{\partial t} = T . \text{ ſin. } p.$$

$$\text{II. } \frac{A \partial y}{\partial t} = - T . \text{ coſ. } p.$$

$$\text{III. } \frac{B \partial x'}{\partial t} = - T . \text{ ſin. } p + T' . \text{ ſin. } q.$$

$$\text{IV. } \frac{B \partial y'}{\partial t} = T . \text{ coſ. } p - T' . \text{ coſ. } q,$$

ex quarum binis prioribus deducitur $\frac{\partial x}{\partial y} = - \text{tang. } p$, tum vero prima cum tertia praebet

$$\frac{A \partial x + B \partial x'}{\partial t} = T' \text{ ſin. } q;$$

ſecun-

secunda autem cum quarta:

$$\frac{A\partial y + B\partial y'}{\partial t} = - T' \operatorname{cof.} q.$$

Haec igitur aequatio per illam diuisa dat

$$\frac{A\partial x + B\partial x'}{A\partial y + B\partial y'} = - \operatorname{tang.} q;$$

sicque ipsae tensiones T et T' e calculo sunt elisae.

§. 6. Nunc igitur loco x' et y' valores ante datos substituamus, et aequationes a tensionibus T et T' liberatae erunt

$$\text{I. } \frac{\partial x}{\partial y} = - \operatorname{tang.} p;$$

$$\text{II. } \frac{(A+B)\partial x + B\, a\, \partial p \operatorname{cof.} p}{(A+B)\partial y + B\, a\, \partial p \operatorname{fin.} p} = - \operatorname{tang.} q;$$

cum quibus aequationibus coniungi oportet supra inuentam

$$y - a \operatorname{cof.} p - b \operatorname{cof.} q = 0.$$

§. 7. Tota igitur nostri problematis solutio perducta est ad tres istas aequationes, in quibus adhuc continentur quatuor quantitates variabiles, binae scilicet coordinatae principales x et y cum binis angulis p et q, quarum ergo ternas per quartam determinare licebit. Ex prima autem commodissime definimus $\partial x = - \partial y \operatorname{tang.} p$, qui valor in secunda substitutus dat

$$- \frac{(A+B)\partial y \operatorname{tang.} p + B\, a\, \partial p \operatorname{cof.} p}{(A+B)\partial y + B\, a\, \partial p \operatorname{fin.} p} = - \operatorname{tang.} q$$

quae reducitur ad hanc formam:

$$(A+B)\partial y(\operatorname{tang.} p - \operatorname{tang.} q) = B\, a\, \partial p(\operatorname{fin.} p \operatorname{tang.} q + \operatorname{cof.} p),$$

hincque porro ad istam:

$$(A+B)\partial y \operatorname{fin.}(q-p) = B\, a\, \partial p \operatorname{cof.} p \operatorname{cof.}(q-p),$$

ideoque

$$\partial y = \frac{B\, a\, \partial p \operatorname{cof.} p \operatorname{cof.}(q-p)}{(A+B)\operatorname{fin.}(q-p)}.$$

At

At vero ex tertia aequatione eſt $y = a \cos. p + b \cos. q$, vnde fit $\partial y = - a \partial p \sin. p - b \partial q \sin. q$, ex quo valore naſcitur haec aequatio:

$$\frac{B a \partial p \cos. p \cos. (q - p)}{(A + B) \sin. (q - p)} = \frac{B a \partial p \cos. p}{(A + B) \tang. (q - p)}$$
$$= - a \partial p \sin. p - b \partial q \sin. q.$$

Hic autem non liquet quomodo reſolutio ſit inſtituenda.

Problema II.

Tab. I. Fig. 8. *Si filo tria corpuscula A, B, C fuerint alligata, eiusque terminus D per lineam rectam I O protrahatur, inueſtigare curuas, quas ſingula corpuscula deſcribent.*

Solutio.

§. 8. Vocentur fili portiones $AB = a$, $BC = b$, $CD = c$, ac demiſſis ad rectam IO perpendiculis AP, BQ, CR ponantur coordinatae:

$$IP = x, \quad IQ = x', \quad IR = x'';$$
$$PA = y, \quad QB = y', \quad RC = y'';$$

tum vero ſtatuantur anguli $PAB = p$, $QBC = q$, $RCD = r$, vnde ſtatim fluunt ſequentes relationes:

$$x' - x = a \sin. p, \quad x'' - x' = b \sin. q,$$
$$y - y' = a \cos. p, \quad y' - y'' = b \cos. q,$$

eſtque $y'' = a \cos. r$, hincque

$$y' = b \cos. q + c \cos. r \text{ et}$$
$$y = a \cos. p + b \cos. q + c \cos. r.$$

§. 9. Pro motu nunc definiendo denotent litterae T, T′ et T″ tenſiones portionum fili AB, BC et CD, ac per hypotheſin ſtabilitam habebimus ſequentes aequationes:

$$\frac{A\partial x}{\partial t} = \text{T fin.}\, p,$$
$$\frac{A\partial y}{\partial t} = -\text{T cof.}\, p,$$
$$\frac{B\partial x'}{\partial t} = -\text{T fin.}\, p + \text{T}' \text{fin.}\, q,$$
$$\frac{B\partial y'}{\partial t} = \text{T cof.}\, p - \text{T}' \text{cof.}\, q,$$
$$\frac{C\partial x''}{\partial t} = -\text{T}' \text{fin.}\, q + \text{T}'' \text{fin.}\, r$$
$$\frac{C\partial y''}{\partial t} = +\text{T}' \text{cof.}\, q - \text{T}'' \text{cof.}\, r.$$

Hinc autem formentur fequentes combinationes:

$$\frac{A\partial x + B\partial x'}{\partial t} = \text{T}' \text{fin.}\, q$$
$$\frac{A\partial y + B\partial y'}{\partial t} = -\text{T}' \text{cof.}\, q$$
$$\frac{A\partial x + B\partial x' + C\partial x''}{\partial t} = \text{T}'' \text{fin.}\, r,$$
$$\frac{A\partial y + B\partial y' + C\partial y''}{\partial t} = -\text{T}'' \text{cof.}\, r.$$

§. 10. Ex his iam aequationibus facile eliminantur tenfiones T, T′ et T″, quippe quae funt incognitae, nihilque ad inftitutum refert eas noffe; tum autem ad tres iftas aequationes peruenietur:

I. $\frac{\partial x}{\partial y} = -\text{tang.}\, p;$

II. $\frac{A\partial x + B\partial x'}{A\partial y + B\partial y'} = -\text{tang.}\, q;$

III. $\frac{A\partial x + B\partial x' + C\partial x''}{A\partial y + B\partial y' + C\partial y''} = -\text{tang.}\, r;$

quibus adiungi oportet aequationem iam fupra inuentam

$$y = a\,\text{cof.}\, p + b\,\text{cof.}\, q + c\,\text{cof.}\, r,$$

in quibus aequationibus, fi loco x', x'' et y', y'' fubftituantur valores fupra affignati, inerunt adhuc hae quinque variabiles: x, y, p, q, r, quarum ergo quaternas per quintam definiri oportet.

§. 11. Subftituamus igitur loco x', x'' et y', y'' fuos valores, et cum fit

 $x' =$

$$x' = x + a \sin. p,$$
$$y' = y - a \cos. p,$$
$$x'' = x + a \sin. p + b \sin. q,$$
$$y'' = y - a \cos. p - b \cos. q,$$

quatuor noſtrae aequationes ita ſe habebunt:

$$\text{I. } \frac{\partial x}{\partial y} = -\text{tang. } p,$$

$$\text{II. } \frac{(A+B)\partial x + B a \partial p \cos. p}{(A+B)\partial y + B a \partial p \sin. p} = -\text{tang. } q,$$

$$\text{III. } \frac{(A+B+C)\partial x + (B+C) a \partial p \cos. p + C b \partial q \cos. q}{(A+B+C)\partial y + (B+C) a \partial p \sin. p + C b \partial q \sin. q} = -\text{tang. } r,$$

$$\text{IV. } y = a \cos. p + b \cos. q + c \cos. r,$$

vbi ex vltima habetur

$$\partial y = -a \partial p \sin. p - b \partial q \sin. q - c \partial r \sin. r$$

ſicque ſolutio noſtri problematis a reſolutione harum aequationum pendet.

§. 12. Cum ex prima harum aequationum ſit $\partial x = -\partial y \,\text{tang.}\, p$, ſubſtituamus hunc valorem in reliquis, vt tantum tres nobis remaneant aequationes, quae erunt:

$$\text{I. } \frac{-(A+B)\partial y \,\text{tang.}\, p + B a \partial p \cos. p}{(A+B)\partial y + B a \partial p \sin. p} = -\text{tang. } q,$$

$$\text{II. } \frac{-(A+B+C)\partial y \,\text{tang.}\, p + (B+C) a \partial p \cos. p + C b \partial q \cos. q}{(A+B+C)\partial y + (B+C) a \partial p \sin. p + C b \partial q \sin. q} = -\text{tang. } r,$$

$$\text{III. } y = a \cos. p + b \cos. q + c \cos. r,$$

priores autem duae aequationes euolutae euadent

$$(A+B)\partial y(\text{tang.}\, q - \text{tang.}\, p) + B a \partial p(\cos. p + \sin. p \,\text{tang.}\, q) = 0,$$

$$(A+B+C)\partial y(\text{tang.}\, r - \text{tang.}\, p) + (B+C) a \partial p(\cos. p + \sin. p \,\text{tang.}\, r)$$
$$+ C b \partial q(\cos. q + \sin. q \,\text{tang.}\, r) = 0,$$

vbi ſi loco ∂y ſcriberemus eius valorem

$$-a \partial p \sin. p - b \partial q \sin. q - c \partial r \sin. r$$

nan-

nancifceremur duas aequationes inter ternos angulos p, q, r quorum binos per tertium definire oportebit.

§. 13. Quemadmodum autem has duas aequationes vlterius tractari conueniat multo minus patet quam in problemate praecedente, quam ob rem fuperfluum foret hanc inueftigationem ad plura corpuscula filo noftro alligata extendere; ita vt hoc negotium penitus abrumpere cogamur.
